Modelo Dinâmico de Manipuladores

Formulação de Lagrange-Euler

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Introdução

$$\tau = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$$

- $L = K - P$ é denominado Lagrangeano
- K é a energia cinética

$$K = \sum_{i=1}^{n} K_i \quad , \quad K_i = \frac{1}{2} m_i V_i^T V_i = \frac{1}{2} m_i \operatorname{Tr}(V_i V_i^T)$$

- P é a energia potencial

$$P = \sum_{i=1}^{n} P_i \quad , \quad P_i = -m_i g^{T\,0} P_{ci} = -m_i g^{T\,0} T_i^{\,i} P_{ci}$$

Velocidade do Centro de Massa

$$ {}^0V_i = \frac{d}{dt}{}^0P_{ci} = \frac{d}{dt}\left({}^0T_i{}^iP_{ci}\right) = \frac{d}{dt}{}^0T_i{}^iP_{ci} $$

$$ {}^0V_i = \left({}^0\dot{T}_1{}^1T_i + {}^0T_1{}^1\dot{T}_2{}^2T_i + \cdots + {}^0T_{i-1}{}^{i-1}\dot{T}_i\right){}^iP_{ci} $$

$$ {}^0V_i = \left(\frac{\partial^0T_1}{\partial q_1}\dot{q}_1{}^1T_i + {}^0T_1\frac{\partial^1T_2}{\partial q_2}\dot{q}_2{}^2T_i + \cdots + {}^0T_{i-1}\frac{\partial^{i-1}T_i}{\partial q_i}\dot{q}_i\right){}^iP_{ci} $$

$$ {}^0V_i = \left(\sum_{j=1}^{i}\frac{\partial^0T_i}{\partial q_j}\dot{q}_j\right){}^iP_{ci} $$

Velocidade do Centro de Massa

- Das convenções de Denavit-Hartenberg:

$$
{}^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Velocidade do Centro de Massa

- para junta rotacional:

$$\frac{\partial^{i-1}T_i}{\partial\theta_i} = \begin{bmatrix} -\sin\theta_i & -\cos\alpha_i\cos\theta_i & \sin\alpha_i\cos\theta_i & -a_i\sin\theta_i \\ \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial^{i-1}T_i}{\partial q_i} = Q_i{}^{i-1}T_i$$

$$Q_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Velocidade do Centro de Massa

- Para junta prismática:

$$\frac{\partial^{i-1}T_i}{\partial d_i} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial^{i-1}T_i}{\partial q_i} = Q_i{}^{i-1}T_i$$

$$Q_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Velocidade do Centro de Massa

Pode-se agora calcular

$$U_{ij} = \frac{\partial {}^0T_i}{\partial q_j} = \begin{cases} {}^0T_{j-1} Q_j {}^{j-1}T_i & \text{, para } j \leq i \\ 0 & \text{, para } j > i \end{cases}$$

de onde pode-se obter

$${}^0V_i = \left(\sum_{j=1}^{i} U_{ij} \dot{q}_j \right) {}^iP_{ci}$$

Energia Cinética dos Elos

$$dK_i = \frac{1}{2}\operatorname{Tr}\left({}^0V_i{}^0V_i^T\right) dm_i$$

$$\begin{aligned}
dK_i &= \frac{1}{2}\operatorname{Tr}\left[\sum_{p=1}^{i} U_{ip}\dot{q}_p{}^iP_{ci}\left(\sum_{r=1}^{i} U_{ir}\dot{q}_r{}^iP_{ci}\right)^T\right] dm_i \\
&= \frac{1}{2}\operatorname{Tr}\left[\sum_{p=1}^{i}\sum_{r=1}^{i} U_{ip}\dot{q}_p{}^iP_{ci}{}^iP_{ci}^T\dot{q}_r U_{ir}^T\right] dm_i \\
&= \frac{1}{2}\operatorname{Tr}\left[\sum_{p=1}^{i}\sum_{r=1}^{i} U_{ip}\left({}^iP_{ci}{}^iP_{ci}^T dm_i\right) U_{ir}^T\dot{q}_p\dot{q}_r\right]
\end{aligned}$$

Energia Cinética dos Elos

- U_{ij} e $\dot{q}_i$ são independentes da distribuição de massa do elo i

$$K_i = \int dK_i = \frac{1}{2}\mathrm{Tr}\left[\sum_{p=1}^{i}\sum_{r=1}^{i} U_{ip}\int {}^iP_{ci}\,{}^iP_{ci}^T dm_i U_{ir}^T \dot{q}_p \dot{q}_r\right]$$

Definindo-se

$$J_i = \int {}^iP_{ci}\,{}^iP_{ci}^T dm_i = \begin{bmatrix} \int x_i^2 dm_i & \int x_i y_i dm_i & \int x_i z_i dm_i & \int x_i dm_i \\ \int x_i y_i dm_i & \int y_i^2 dm_i & \int y_i z_i dm_i & \int y_i dm_i \\ \int x_i z_i dm_i & \int y_i z_i dm_i & \int z_i^2 dm_i & \int z_i dm_i \\ \int x_i dm_i & \int y_i dm_i & \int z_i dm_i & \int dm_i \end{bmatrix}$$

Energia Cinética dos Elos

$$
\begin{aligned}
K &= \sum_{i=1}^{n} K_i = \frac{1}{2}\sum_{i=1}^{n} \mathrm{Tr}\left(\sum_{p=1}^{i}\sum_{r=1}^{i} U_{ip} J_i U_{ir}^T \dot{q}_p \dot{q}_r\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{p=1}^{i}\sum_{r=1}^{i} \mathrm{Tr}\left(U_{ip} J_i U_{ir}^T\right) \dot{q}_p \dot{q}_r
\end{aligned}
$$

Tensor de Inércia

$$I_{ij} = \int \left[\delta_{ij} \left(\sum_k x_k^2 \right) - x_i x_j \right] dm$$

$$J_i = \begin{bmatrix} \frac{-I_{xx}+I_{yy}+I_{zz}}{2} & I_{xy} & I_{xz} & m_i x_{ci} \\ I_{xy} & \frac{I_{xx}-I_{yy}+I_{zz}}{2} & I_{yz} & m_i y_{ci} \\ I_{xz} & I_{yz} & \frac{I_{xx}+I_{yy}-I_{zz}}{2} & m_i z_{ci} \\ m_i x_{ci} & m_i y_{ci} & m_i z_{ci} & m_i \end{bmatrix}$$

Energia Potencial dos Elos

$$P = -\sum_{i=1}^{n} m_i g^{T\,0} T_i^{\,i} P_{ci}$$

Lagrangeano

$$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{p=1}^{i}\sum_{r=1}^{i}\mathrm{Tr}\left(U_{ip}J_iU_{ir}^T\right)\dot{q}_p\dot{q}_r + \sum_{i=1}^{n} m_i g^{T0}T_i^{\,i}P_{ci}$$

Torque

$$\tau_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=i}^{n}\sum_{k=1}^{j} \mathrm{Tr}\left(U_{jk}J_jU_{ji}^T\right)\dot{q}_k$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=i}^{n}\sum_{k=1}^{j} \mathrm{Tr}\left(U_{jk}J_jU_{ji}^T\right)\ddot{q}_k + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} \frac{\partial}{\partial q_m}\mathrm{Tr}\left(U_{jk}J_jU_{ji}^T\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=i}^{n}\sum_{k=1}^{j} \mathrm{Tr}\left(U_{jk}J_jU_{ji}^T\right)\ddot{q}_k + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} \mathrm{Tr}\left(\frac{\partial U_{jk}}{\partial q_m}J_jU_{ji}^T\right)\dot{q}_k$$

Torque

Definindo

$$U_{jkm} = \frac{\partial U_{jk}}{\partial q_m} = \begin{cases} {}^0T_{k-1}Q_k{}^{k-1}T_{m-1}Q_m{}^{m-1}T_j & , j \geq m \geq k \\ {}^0T_{m-1}Q_m{}^{m-1}T_{k-1}Q_k{}^{k-1}T_j & , j \geq k \geq m \\ 0 & , j < k \text{ ou } j < m \end{cases}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=i}^{n}\sum_{k=1}^{j} \operatorname{Tr}\left(U_{jk}J_jU_{ji}^T\right)\ddot{q}_k + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} \operatorname{Tr}\left(U_{jkm}J_jU_{ji}^T\right)\dot{q}_k\dot{q}_m$$

Torque

$$\frac{\partial L}{\partial q_i} = \sum_{j=i}^{n} m_j g^T U_{ji}{}^j P_{cj}$$

$$\begin{aligned}\tau_i \;&=\; \sum_{j=i}^{n}\sum_{k=1}^{j} \mathrm{Tr}\left(U_{jk} J_j U_{ji}^T\right)\ddot{q}_k \\ &+\; \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} \mathrm{Tr}\left(U_{jkm} J_j U_{ji}^T\right)\dot{q}_k\dot{q}_m \\ &-\; \sum_{j=i}^{n} m_j g^T U_{ji}{}^j P_{cj}\end{aligned}$$

Torque

$$\tau_i = \sum_{k=1}^{n} M_{ik}\ddot{q}_k + \sum_{k=1}^{n}\sum_{m=1}^{n} V_{ikm}\dot{q}_k\dot{q}_m + G_i$$

$$M_{ik} = \sum_{j=\max(i,k)}^{n} \mathrm{Tr}\left(U_{jk}J_jU_{ji}^T\right)$$

$$V_{ikm} = \sum_{j=\max(i,k,m)}^{n} \mathrm{Tr}\left(U_{jkm}J_jU_{ji}^T\right)$$

$$G_i = \sum_{j=i}^{n} -m_j g^T U_{ji}{}^j P_{cj}$$

Torque

- na forma matricial:

$$\tau = M(q)\ddot{q} + V(q, \dot{q}) + G(q)$$

Modelo no Espaço de Estados

$$\ddot{q} = M^{-1}(q)\left[\tau - V(q,\dot{q}) - G(q)\right]$$

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$u = \tau$$

$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1)u$$

$$f_2(x_1, x_2) = -M^{-1}(x_1)\left(V(x_1, x_2) + G(x_1)\right)$$

$$g_2(x_1) = M^{-1}(x_1)$$

Modelo no Espaço de Estados

- na forma afim:

$$\dot{x} = f(x) + g(x)u$$

sendo

$$f(x) = \begin{bmatrix} x_2 \\ f_2(x_1, x_2) \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ g_2(x_1) \end{bmatrix}$$